11:00 - 17

1.5 – Elementary Matrices and a Method for Finding A^{-1}

Definition: Matrices A and B are said to be **row equivalent** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

– useful for developing theory

Definition: A matrix E is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.

Theorem 1.5.1 Row Operations by Matrix Multiplication
If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

#6 An elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A.

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $R_1 \Rightarrow -6R_1$

a. $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

b. $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$ b) $R_2 \Rightarrow R_2 - 4R_1$

c. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

a) A: $R_1 \Rightarrow -6R_1$ yields $\begin{bmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

$EA = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 6 & 12 & -30 & 6 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

b) $EA = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ -7 & 1 & -1 & 21 & 19 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

$A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix} \; R_2 \Rightarrow R_2 - 4R_1 \; \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ -7 & 1 & -1 & 21 & 19 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

Thm 1.5.1

Elem. Row ops:
- add scalar mult. of one row to another
- mult. a row by a nonzero scalar
- Interchange two rows

pf: The case in which E results from multiplying the i^{th} row of I by k.

Let $\vec{e}_1 = [1 \; 0 \; 0 \cdots 0]$, $\vec{e}_2 = [0 \; 1 \; 0 \; \ldots \; 0]$, $\vec{e}_m = [0 \; 0 \; \cdots \; 1]$, $k\vec{e}_i = [0 \; 0 \cdots k \cdots 0]$

$$EA = \begin{bmatrix} \vec{e}_1 A \\ \vec{e}_2 A \\ \vdots \\ k\vec{e}_i A \\ \vdots \\ \vec{e}_m A \end{bmatrix},$$ which results in multiplying the i^{th} row of A by k. ✓

$I \xrightarrow[\text{op}]{\text{one elem. row}} E$

#7 Use the following matrices and find an elementary matrix E that satisfies the stated equation.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix},\ B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix},\ C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix},$$

$R_3 \to R_3 - 2R_1$

$$D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix},\ F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

c. $EA = C$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\cancel{AE = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}$$

Check: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}$ ✓

Theorem 1.5.2 Every elementary matrix is invertible, and the inverse is also an elementary matrix.

pf: Let E be the elementary matrix that results from performing a single row operation on I and let E' be the matrix that results from performing the inverse

operation on I. Then $EE' = I$ and
$E'E = I \Rightarrow E' = E^{-1}$.

Theorem 1.5.3 Equivalent Statements
If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all are true or all false.
a) A is invertible.
b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.

there is ... such that

Pf: $a \Rightarrow b$ A invertible $\Rightarrow \exists A^{-1} \ni AA^{-1} = I$

$A\vec{x} = \vec{0} \Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{0} \Rightarrow \vec{x} = \vec{0}$

If $\vec{x}'$ is any other solution, then (1.4.2d)

$A\vec{x} = \vec{0}$ and $A\vec{x}' = \vec{0}$

$\Rightarrow A\vec{x} - A\vec{x}' = \vec{0} \Rightarrow A(\vec{x} - \vec{x}') = \vec{0}$

$\vec{x} - \vec{x}' = A^{-1}\vec{0} = \vec{0} \Rightarrow \vec{x} = \vec{x}'$. ✓

$b \Rightarrow c$ $A\vec{x} = \vec{0}$ has only the trivial solution so the system contains no free variables. By Thm 1.2.1, rref has n nonzero rows. By Thm 1.4.3, rref is I_n. ✓

$c \Rightarrow d$ rref of A is I_n, so I_n can be obtained from A by a series of elem. row operations. We can express $I_n = (E_r \cdots E_2 E_1) A$ by Thm 1.5.1.

E_i is invertible. $A = E_1^{-1} E_2^{-1} \cdots E_r^{-1} \cancel{I}$,

a product of elem. matrices. ✓

$d \Rightarrow a$ the matrix product $E_r \cdots E_2 E_1$

in the previous step is A^{-1}.

so A is invertible ✓

Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

#15 $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

The inversion algorithm is based on the $c \Rightarrow d$

proof above. Since $I_n = E_r \cdots E_2 E_1 A$,

we have $A^{-1} = (E_r \cdots E_2 E_1) I_n$

Thus the elementary row operations that

reduce A to I will yield A^{-1}

When applied to I. Thus: $[A|I] \rightarrow [I|A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$

$$\begin{array}{rrrrrr} 2 & 7 & 6 & 0 & 1 & 0 \\ -2 & -6 & -6 & -1 & 0 & 0 \\ \hline 0 & 1 & 0 & -1 & 1 & 0 \end{array}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{array}{rrrrrr} 2 & 7 & 7 & 0 & 0 & 1 \\ -2 & -7 & -6 & 0 & -1 & 0 \\ \hline 0 & 0 & 1 & 0 & -1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$R_1 \rightarrow R_1 - 6R_3$

$$\begin{array}{rrrrrr} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 & 6 & -6 \\ \hline 2 & 6 & 0 & 1 & 6 & -6 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$R_1 \rightarrow R_1 - 6R_2$

$$\begin{array}{rrrrrr} 2 & 6 & 0 & 1 & 6 & -6 \\ 0 & -6 & 0 & 6 & -6 & 0 \\ \hline 2 & 0 & 0 & 7 & 0 & -6 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$R_1 \rightarrow \frac{1}{2}R_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

so $A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

#18 $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

#20 Find the inverse of each of the following 4×4 matrices, where k_1, k_2, k_3, k_4, k are all nonzero.

a. $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$

#27 Show that the matrices A and B are row equivalent by finding a sequence of elementary row operations that produces B from A, and then use that result to find a matrix C such that $CA = B$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To obtain B from A:

Corresponding Elementary matrix

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 2 & 1 & 9 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix} = B \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

we have $E_3 E_2 E_1 A = B$

So $C = E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

#23 Express the matrix and its inverse as products of elementary matrices.

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$$

Want $A = E_1 E_2 \cdots E_n$ and $A^{-1} = E_1' E_2' \cdots E_n'$

Reduce A to I_2, keeping track of the ops.

$R_2 \to R_2 + \frac{2}{3} R_1$ — Corresponding Elem. Matrix — Inverse

$$\begin{array}{cc} 2 & 2 \\ -2 & 2/3 \\ \hline 0 & 8/3 \end{array}$$

$$\begin{bmatrix} -3 & 1 \\ 0 & 8/3 \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{3} & 1 \end{bmatrix}$$

$R_1 \to R_1 - \frac{3}{8} R_2$

$$\begin{array}{cc} -3 & 1 \\ 0 & -1 \\ \hline -3 & 0 \end{array} \qquad E_2 = \begin{bmatrix} 1 & -3/8 \\ 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 3/8 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 0 \\ 0 & 8/3 \end{bmatrix}$$

$R_1 \to -\frac{1}{3} R_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & 8/3 \end{bmatrix} \qquad E_3 = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$$

$R_2 \to \frac{3}{8} R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad E_4 = \begin{bmatrix} 1 & 0 \\ 0 & 3/8 \end{bmatrix} \qquad E_4^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 8/3 \end{bmatrix}$$

$$\underbrace{(E_4 E_3 E_2 E_1)}_{A^{-1}} A = I \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3/8 \end{bmatrix} \begin{bmatrix} -1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3/8 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} \Rightarrow A = \begin{bmatrix} 1 & 0 \\ -2/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3/8 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 8/3 \end{bmatrix}$$

#33 Prove that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A.

Let $E_1, E_2, \dots, E_n$ be the elementary matrices that represent the sequence of row operations that reduce A to B.

Then $E_n \cdots E_2 E_1 A = B$.

Each E_i is invertible, so

$$A = (E_n \cdots E_2 E_1)^{-1} B = E_1^{-1} E_2^{-1} \cdots E_n^{-1} B$$

thus there is a sequence of elementary row operations, represented by E_i^{-1} that reduces B to A. ✓